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Magnetic instabilities of a rotating gas

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We explore two types of instability which may develop when a highly conducting gas rotates rapidly in the presence of a radial gravitational force and an azimuthal magnetic field. Beyond a critical radius (equal to twice the isothermal scale height) a decrease of magnetic flux (per unit mass) outwards leads to the appearance of eastward-propagating waves by the mechanism of 'magnetic buoyancy'. Within the critical radius an increase of magnetic flux outwards leads to westward-propagating waves by a totally different mechanism. Provided that the effects of Ohmic dissipation are not too large, either instability may set in for quite modest magnetic flux gradients, even when the magnetic energy of the system is very much smaller than the rotational energy.

1. Introduction

Consider a horizontal layer of perfect gas at rest in the presence of gravity g and a horizontal magnetic field $B(z)$ in the x direction. Let the gas be inviscid and a perfect conductor of both electricity and heat, so that a moving fluid parcel immediately adjusts its temperature to that of its surroundings. The original temperature field is then unaltered by any subsequent motion, and if we take for simplicity an isothermal basic state, the pressure p and density ρ are related throughout that motion by $p = a^2\rho$, where the isothermal sound speed a is constant. The basic balance is magnetostatic:

$$d(p+\tfrac{1}{2}\mu^{-1}B^2)/dz+\rho g = 0, \tag{1.1}$$

where μ denotes the magnetic permeability. If we now consider instead a field distribution $B(y, z)$ in the form of a thin magnetic flux tube at some particular height, the total pressure $p+\frac{1}{2}\mu^{-1}B^2$ must be the same both inside and just outside the tube. The fluid pressure p must therefore be somewhat smaller inside the tube (where $B \neq 0$) than just outside, the same is therefore true of the density, and the tube tends to rise. This mechanism of 'magnetic buoyancy' was pointed out by Parker (1955) and Jensen (1955), and has subsequently been much discussed in connexion with the dynamics of the upper layers of the sun (see, for example, Parker 1976). Note that our assumption here of infinitely fast diffusion of heat has filtered out *conventional* buoyancy forces that would otherwise arise owing to the vertical entropy gradient.

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Gilman (1970) showed that when $B(z)$ decreases continuously with height this mechanism of magnetic buoyancy renders the system unstable to disturbances whose wavenumber component k in the x direction is such that

$$-\frac{g}{a^2}\frac{d}{dz}\log B > k^2. \tag{1.2}$$

The case $k = 0$, for which the motions are two-dimensional and carry flux tubes bodily about without distorting them, requires separate consideration, and Gilman & Cadez (in an appendix to Gilman 1970) showed that such instability is possible only if the magnetic field falls off with height *faster than the density*, i.e.

$$-\frac{d}{dz}\left(\frac{B}{\rho}\right) > 0. \tag{1.3}$$

This criterion may be easily interpreted physically in terms of an elementary interchange argument (appendix A), but a physical explanation of (1.2), and why disturbances with k small but non-zero are the most readily amplified, is more intricate. Essentially a little twisting is helpful, since it permits the flow of fluid down the rising portions of the distorted flux tubes to the sinking portions, which enhances the magnetic buoyancy effect (Parker 1955), while too much twisting (i.e. k too large) results in the restoring forces arising from the 'elasticity' of the field lines outweighing the magnetic buoyancy effects.

The real starting-point for the present investigation is Gilman's extension of this analysis (locally) to a uniformly *rotating* spherical body of gas. He found that the most unstable modes were of short wavelength in the y direction (i.e. northwards), so that only the component of rotation perpendicular to gravity ($\Omega \sin\theta$, where θ is the polar angle) was significant. He examined in detail the particular case $B \propto \rho^{\frac{1}{2}}$ (i.e. Alfvén speed constant with height) and showed that rotation completely stabilizes the system against magnetic buoyancy when

$$V^2/\Omega^2 H^2 < 8\sin^2\theta, \tag{1.4}$$

where $H = a^2/g$ is the isothermal scale height. For astrophysical applications of the theory to the radiative interiors or convective envelopes of stars (see §6), the most important parameter regime appears to be

$$V^2 \ll \Omega^2 r^2 \ll a^2 \lesssim gr, \tag{1.5}$$

where typical values of the Alfvén speed V and radius r are to be taken. Thus (1.4) is usually well satisfied (by many orders of magnitude), except possibly very close indeed to the surface of a star, where $H \ll r$.

The main result of the present paper, in the above context, is that if B decreases with height faster than ρ (which Gilman's B distribution did not) rapid rotation of this kind does *not* stop magnetic buoyancy instability.† In fact we study a somewhat more general *cylindrically symmetric* basic equilibrium state with an azimuthal magnetic field $B(r)$, uniform rotation and a gravitational body force directed normal to the rotation axis. This permits account to be taken of curvature effects, and also allows some (though regrettably few) results to be derived (in §4) on other than a 'local'

† This has also been noted by Professor H. K. Moffatt in §10.7 of his forthcoming monograph *Magnetic Field Generation in Electrically Conducting Fluids*, and by Professor P. H. Roberts and Professor K. Stewartson in an unpublished earlier version of the paper by them referenced and briefly discussed in §6.

basis. In the rapidly rotating parameter regime described by (1.5) the system is unstable provided only that

$$r^2\left(\frac{2}{r}-\frac{g}{a^2}\right)\frac{d}{dr}\left[\log\left(\frac{B}{\rho r}\right)\right] > 1, \tag{1.6}$$

and non-axisymmetric waves which travel azimuthally with a slow angular propagation speed

$$\left(\frac{g}{a^2}-\frac{2}{r}\right)\frac{V^2}{2\Omega r} \tag{1.7}$$

then spontaneously amplify with a growth rate comparable to their frequency $\sim V^2/\Omega r^2$. A critical radius, given (implicitly, since g will of course vary with r in a prescribed way) by

$$r_c = 2a^2/g, \tag{1.8}$$

thus emerges naturally from the stability analysis, and is in fact that radius at which an isolated magnetic flux ring would sit in equilibrium, its tendency to rise by magnetic buoyancy being exactly balanced by its tendency to collapse owing to the 'magnetic hoop stress' $B^2/\mu r$ arising from the curvature of its magnetic field lines.† The local stability properties of the system evidently depend crucially on whether the region under discussion lies inside or outside the critical radius. Outside the critical radius a modest decrease of $B/\rho r$ outwards leads to magnetic buoyancy instability in the form of eastward-propagating waves. Inside the critical radius a modest increase of $B/\rho r$ outwards leads to westward-propagating waves by a totally different instability mechanism.

It is important to note that the instability criterion (1.6), valid in the parameter range (1.5), is independent of the rotation rate Ω, so that the system as it stands cannot be stabilized by rotation, however rapid. To be sure, the growth rates are of order $V^2/\Omega r^2$ and thus diminish as Ω increases, but to suppress the instability altogether by rapid rotation requires the inclusion of Ohmic dissipation in the model (§5). In §6 we discuss very briefly the more subtle role that diffusive effects may play if, in addition, the thermal diffusivity κ is given a large, but finite, value so that the gas is no longer a perfect thermal conductor.

2. Mathematical formulation

Given the assumptions of §1 the basic hydromagnetic equations for our problem are

$$\rho(\partial \mathbf{u}/\partial t+\mathbf{u}.\nabla\mathbf{u}) = -\nabla p+\mu^{-1}(\nabla\wedge\mathbf{B})\wedge\mathbf{B}+\rho\mathbf{g}^*, \tag{2.1}$$

$$\partial\mathbf{B}/\partial t = \nabla\wedge(\mathbf{u}\wedge\mathbf{B}), \tag{2.2}$$

† The pressure in the flux ring is lower than that of its surroundings by $B^2/2\mu$, its density is therefore lower by $B^2/2\mu a^2$, so the magnetic buoyancy force is $gB^2/2\mu a^2$. This is in balance with $B^2/\mu r$ when $r = r_c$. Placed inside r_c such an isolated flux ring would collapse, while if placed outside $r = r_c$ the curvature of the magnetic field lines would be too weak to prevent the ring from rising under its own magnetic buoyancy. This significance of the critical radius was apparently first recognized by Weiss (1964) (but see also Jensen 1955), and has been discussed at length in an extensive study of magnetic buoyancy instability, mainly in the case of zero or low rotation, by Cadez (1974).

$$\nabla . \mathbf{B} = 0, \tag{2.3}$$

$$\partial\rho/\partial t + \nabla . (\rho \mathbf{u}) = 0, \tag{2.4}$$

$$p = a^2 \rho. \tag{2.5}$$

Here ρ denotes the fluid density, $\mathbf{u}$ the Eulerian velocity, p the pressure, μ the magnetic permeability, $\mathbf{B}$ the magnetic field, $\mathbf{g}^*$ gravity and t time. We shall study the stability of the equilibrium configuration in which the fluid rotates with uniform angular velocity Ω and is at uniform temperature T, so that the isothermal sound speed $a = (RT)^{\frac{1}{2}}$ (where R is the gas constant) is constant. Referring all quantities to a set of cylindrical polar co-ordinates (r, θ, z), the basic magnetic field, pressure and density distributions

$$\mathbf{B} = \{0, B(r), 0\}, \quad p = p(r), \quad \rho = \rho(r) \tag{2.6}$$

represent an exact solution of the basic equations in the presence of a radial gravitational force $\mathbf{g}^* = \{-g^*(r), 0, 0)$ provided that

$$\frac{d}{dr}\left(p + \frac{B^2}{2\mu}\right) + \frac{B^2}{\mu r} + \rho(g^* - \Omega^2 r) = 0 \tag{2.7}$$

and, of course,

$$p(r) = a^2 \rho(r). \tag{2.8}$$

If we slightly disturb the system the linearized forms of (2.1)–(2.5) admit solutions in which (by virtue of the equilibrium configuration) all perturbation quantities ϕ may be written as

$$\phi = \mathscr{R}[\hat{\phi}(r) \exp i(m\theta + nz - \sigma t)], \tag{2.9}$$

where m, n and σ are constants. The last of these (which may be complex) represents the frequency of oscillation as seen by an inertial observer, and it proves convenient to define a Doppler-shifted frequency

$$\omega \equiv \sigma - m\Omega, \tag{2.10}$$

which is that measured by an observer rotating with angular velocity Ω. It is convenient also to define the Alfvén speed

$$V(r) \equiv B(r)/\{\mu\rho(r)\}^{\frac{1}{2}}, \tag{2.11}$$

the magneto-acoustic speed

$$c(r) \equiv [V^2(r) + a^2]^{\frac{1}{2}}, \tag{2.12}$$

the 'apparent' gravity $g \equiv g^* - \Omega^2 r$ and the local azimuthal wavenumber $k(r) \equiv m/r$. After a great deal of algebra, which we omit here, it is possible to eliminate all perturbation variables in favour of the radial velocity component $\hat{u}_r$, which satisfies the equation

$$\left[\left(\rho c^2 + \frac{P}{\lambda}\right) r\hat{u}_r'\right]' + rF\hat{u}_r = 0, \tag{2.13}$$

where a prime denotes differentiation with respect to r.

Here P and λ are comparatively simple functions of r given by

$$\lambda(r) \equiv \omega^4 - (k^2 + n^2) c^2 \omega^2 + V^2 k^2 a^2 (k^2 + n^2), \tag{2.14}$$

and

$$\rho^{-1} P(r) \equiv n^2 V^2 c^2 (\omega^2 - a^2 k^2) + a^2 n^2 V^2 \omega^2 + a^4 \{(k^2 + n^2) \omega^2 - V^2 k^4\}. \tag{2.15}$$

Unfortunately F is very complicated indeed, being given by

$$F(r) \equiv \rho(\omega^2 - V^2k^2) + \left(\rho' + \frac{\rho}{r}\right)\left(g - \frac{a^2}{r}\right) + \frac{a^2}{r^2}(r^2\rho')' + \frac{1}{r^2}\left[r^2\left(\frac{B^2}{2\mu}\right)'\right]' + \left(\frac{Q}{\lambda}\right)' - \frac{\rho E}{\lambda}, \quad (2.16)$$

where the subsidiary parameters Q and E are defined by

$$\rho^{-1}Q(r) \equiv 2\Omega\omega k a^2(\omega^2 - V^2k^2) - \frac{n^2V^2}{r}(c^2\omega^2 - V^2k^2a^2) - \left(g - \frac{a^2}{r}\right)\{[a^2(k^2+n^2) + n^2V^2]\,\omega^2 - a^2k^2V^2(k^2+n^2)\} \quad (2.17)$$

and

$$E(r) \equiv -4\Omega\omega k\left(g - \frac{a^2}{r}\right)\{\omega^2 - V^2(k^2+n^2)\} - \frac{2\Omega\omega k a^2}{r}(\omega^2 - V^2k^2) - \frac{4\Omega\omega k n^2 a^2 V^2}{r} + 4\Omega^2\omega^2(\omega^2 - k^2V^2 - n^2c^2) + (\omega^2 - V^2k^2)(k^2+n^2)\left(g - \frac{a^2}{r}\right)^2 + \frac{1}{r}\left(g - \frac{a^2}{r}\right)\{3n^2V^2\omega^2 + a^2(k^2+n^2)(\omega^2 - V^2k^2)\} + \frac{n^2V^2}{r^2}[a^2\omega^2 + 2V^2(\omega^2 - a^2k^2)]. \quad (2.18)$$

An eigenvalue problem for ω results if we append suitable boundary conditions to (2.13), e.g. $\hat{u}_r = 0$ at $r = r_1$ and $r = r_2$. Such is the complexity of the coefficients of the differential equation, however, that only a few general results have been obtained, and these will be reported in §4. The next two sections will deal with axisymmetric and non-axisymmetric disturbances, respectively, almost entirely on the basis of a 'local' or 'narrow-gap' analysis in which we confine attention to the neighbourhood of a particular radius, r_0 say. We assume that all variables in the equilibrium state vary with radius by a factor of $O(1)$ over a radial distance $O(r_0)$, but that perturbation variables (such as u_r), while having azimuthal wavelengths $2\pi/k$ also $O(r_0)$, have very short radial and axial wavelengths, $O(d)$ say, where $d \ll r_0$. Thus with error $O(d/r_0)$ we may replace (2.13) by

$$\left(\rho c^2 + \frac{P}{\lambda}\right)\hat{u}_r'' + F\hat{u}_r = 0, \quad (2.19)$$

and by expanding the coefficients in a Taylor series about $r = r_0$ they may be replaced by their values *at* $r = r_0$ with the same error $O(d/r_0)$. The equation then admits solutions of the form $\hat{u}_r \propto \sin l(r - r_0)$, where l is the local radial wavenumber. Such a solution would, incidentally, be compatible with narrowly spaced cylindrical boundaries at $r = r_0$ and $r = r_0 + \pi/l$. We then have the extremely complicated algebraic equation

$$-l^2\left(\rho c^2 + \frac{P}{\lambda}\right)_{r=r_0} + F_{r=r_0} = 0 \quad (2.20)$$

to deal with, and confine attention to locating (approximately) those roots ω which as $d/r_0 \to 0$ do not depend on d itself. It is our hope that the physical properties of such modes are relatively insensitive to the 'short-wavelength' or 'local' approximations which we have been forced to make. Thus by taking the limit $n \to \infty$ in (2.14)–(2.18), P, λ, Q and E all simplify considerably and become $O(n^2)$ while F and P/λ become

$O(1)$. To retain the first term of (2.20) we formally let $l \to \infty$ by keeping l/n constant, $O(1)$. We thereby obtain the dispersion relationship

$$A_0\omega^4 + A_2\omega^2 + A_3\omega + A_4 = 0, \tag{2.21}$$

where

$$A_0 = (1 + l^2/n^2)c^2, \tag{2.22}$$

$$-A_2 = V^2\left(g - \frac{2a^2}{r}\right)\left[\log\left(\frac{B}{\rho r}\right)\right]' + 4\Omega^2c^2 + V^2k^2(c^2+a^2)\left(1+\frac{l^2}{n^2}\right), \tag{2.23}$$

$$A_3 = 4\Omega V^2k(g - 2a^2/r), \tag{2.24}$$

$$A_4 = V^4k^2\left\{\left(g - \frac{2a^2}{r}\right)[\log(Br)]' + a^2k^2\left(1+\frac{l^2}{n^2}\right)\right\}, \tag{2.25}$$

the values of these quantities *at* $r = r_0$ being understood. Since the roots of (2.21) do not depend on l or n individually but only on their ratio, they must be adequate approximations to some of the roots of the original equation (2.20) for sufficiently small values of d/r_0. Just *how* small d/r_0 has to be will in general depend on the root in question, and though vital to the calculations of second-order quantities such as the helicity (which we shall present in a later paper) it will not concern us here.

3. Axisymmetric instability

When $k = 0$, (2.21) simplifies considerably, since $A_3 = A_4 = 0$, and we obtain

$$\omega^2 = \left\{\left(g - \frac{2a^2}{r}\right)\frac{V^2}{c^2}\left[\log\left(\frac{B}{\rho r}\right)\right]' + 4\Omega^2\right\}\frac{n^2}{l^2+n^2}. \tag{3.1}$$

It immediately follows that the system is unstable to axisymmetric disturbances if, and only if,

$$\left(\frac{2a^2}{r} - g\right)\frac{V^2}{c^2}\left[\log\left(\frac{B}{\rho r}\right)\right]' > 4\Omega^2. \tag{3.2}$$

This is an elementary extension (also noted in a spherical-geometry context by Cadez 1974) of a result due to Schubert (1968) to include rotation, which appears in a very simple and stabilizing way. As Schubert noted, whether given magnetic field and density distributions tend to destabilize the system or not depends very much on whether the region being discussed lies inside or outside the critical radius $r_c = 2a^2/g$.

In order to understand the energetic aspects of the instability imagine two thin rings of fluid centred on the rotation axis, at radii r_0 and $r_0 + \delta r$ respectively, which are to be interchanged in such a way that each assumes the volume vacated by the other. During the interchange each conserves its own mass and magnetic flux (the fluid is perfectly electrically conducting and therefore Alfvén's theorem holds) and remains at temperature T. These conditions suffice for the calculation of the net changes in gravitational potential, magnetic and internal (elastic) energy after the interchange, which are respectively given by

$$\Delta E_{\text{pot}} = gAK, \tag{3.3}$$

$$\Delta E_{\text{mag}} = \left(g - \frac{2a^2}{r}\right)\frac{a^2}{c^2}AK, \tag{3.4}$$

$$\Delta E_{\text{int}} = -\left(g + \frac{2V^2}{r}\right)\frac{a^2}{c^2}AK. \tag{3.5}$$

We have here introduced for convenience

$$K \equiv \rho\xi_0(\delta r)^2, \quad A \equiv \frac{V^2}{c^2}\left[\log\left(\frac{B}{\rho r}\right)\right]', \tag{3.6}$$

where ξ_0 is the original volume of the ring that started at $r = r_0$. The details of the calculation are rather heavy and are postponed to appendix B. It is physically instructive to compare the *sum* $(-2a^2/r)AK$ of the magnetic and internal energy changes with the potential energy change (3.3), for this brings out the importance of the critical radius from another point of view. Evidently in circumstances when (in the absence of rotation) (3.2) predicts instability our interchange calculation displays a net *release* of energy.

Outside the critical radius a distribution of $B/\rho r$ decreasing outwards promotes instability by magnetic buoyancy; the instability releases magnetic and gravitational potential energy but absorbs internal (elastic) energy. The simplest illustrative case is well beyond the critical radius, where curvature effects are unimportant [formally let $r \to \infty$ in (3.3)–(3.6)], and the contributions (3.4) and (3.5) cancel, so that the instability occurs simply to release gravitational potential energy stored in the basic state.

Inside the critical radius a distribution of $B/\rho r$ *increasing* outwards promotes instability of a quite different kind; the instability releases magnetic and elastic energy but absorbs gravitational potential energy. The simplest illustrative case is when the isothermal sound speed is very large, for instability then essentially releases only magnetic energy due to the 'hoop stress' arising from the field-line curvature. This instability mechanism is possible in an incompressible fluid of constant density (see, for example, Acheson 1972), while magnetic buoyancy is not.

Equation (3.2) shows that uniform rotation of the whole system has a stabilizing influence, and it in fact totally suppresses both kinds of *axisymmetric* instability in the parameter regime (1.5) of most astrophysical interest. This is because the amount of available energy of the kind discussed above is then diminutive compared with the amount of work needed to effect the increase in rotational kinetic energy

$$\Delta E_{\text{kin}} = 4\Omega^2 K \tag{3.7}$$

implied by the fact that each ring must conserve its angular momentum as it moves. This is because no axial torque acts on either ring, which in turn follows because (i) the gravitational force is purely radial, (ii) the gas is inviscid, and (iii) there is no component of the Lorentz force $(\nabla \wedge \mathbf{B}) \wedge \mathbf{B}$ in the azimuthal direction because axisymmetric disturbances do no twist the field lines and $\mathbf{B}$ itself therefore remains in that direction.

4. Slow wavy instabilities at rapid rotation speeds

Drawing on the analogy between some of the results so far (particularly those valid inside the critical radius) and those for the corresponding stability problem for an incompressible liquid of constant density (Acheson 1972), we are led to explore the possibility that at rapid rotation speeds satisfying (1.5) instability may occur in the form of *non-axisymmetric waves* of low frequency $\omega \sim V^2/\Omega r^2$. If we seek such solutions to (2.21) in the parameter regime (1.5) the coefficient A_2 simplifies enormously to

$-4\Omega^2c^2$, and the term $A_0\omega^4$ is discarded altogether. The resulting quadratic is easily solved to give, on using (2.7),

$$\frac{2\Omega\omega}{V^2k} = \frac{g}{a^2} - \frac{2}{r} \pm i\left\{\left(\frac{2}{r} - \frac{g}{a^2}\right)\left[\log\left(\frac{B}{\rho r}\right)\right]' - \left(1 + \frac{l^2}{n^2}\right)k^2\right\}^{\frac{1}{2}}, \tag{4.1}$$

and instability thus occurs in the assumed way for modes satisfying

$$\left(\frac{2}{r} - \frac{g}{a^2}\right)\left[\log\left(\frac{B}{\rho r}\right)\right]' > \left(1 + \frac{l^2}{n^2}\right)k^2. \tag{4.2}$$

Comparing (4.2) and (3.2) we see that exactly the same remarks about the types of configuration which promote instability (i.e. $(B/\rho r)' < 0$ outside the critical radius, $(B/\rho r)' > 0$ inside) still apply: the essential difference is simply that the enormous stabilizing term due to rotation, $4\Omega^2$, has completely disappeared and been replaced by a much smaller one (by a factor of order V^2/Ω^2r^2) on the right-hand side of (4.2). Evidently the azimuthal component of the Lorentz force brought about by the twisting of the (originally) azimuthal magnetic field lines breaks, most effectively, the constraint that each ring (which now gets slightly distorted, of course, during its motion) must conserve its angular momentum. In place of the enormous amount of work needed to effect the implied increase in rotational kinetic energy (3.7), all that is now needed is a comparatively modest amount, represented by the right-hand side of (4.2), to twist the azimuthal field lines against the resistance of their own 'magnetic hoop stress', which increases with the amount of twisting, whence the factor k^2.

Thus an $O(1)$ gradient (of appropriate sign) of the magnetic flux per unit mass $B/\rho r$ is quite sufficient for the spontaneous amplification of low frequency waves with azimuthal wavenumber $m \sim 1$, i.e. $k \sim r_0^{-1}$. Their growth rates are typically comparable with the frequency, $\sim V^2/\Omega r^2$. The direction of propagation evidently depends, like so much else here (!), on whether the region under discussion lies inside or outside the critical radius. Inside the critical radius we find from (4.1) that $\Omega k\omega_R < 0$, and the slow waves propagate westwards as in the incompressible, constant density case (Acheson 1972). In complete contrast to this, *outside* the critical radius amplifying slow waves propagate *eastwards*.

We have already remarked that the complexity of the coefficients in (2.13) makes stability analysis on anything but a 'local' basis extremely difficult, but this eastward propagation of slow amplifying waves outside the critical radius is one result which can be established globally, as we now show. By restricting attention to the slow waves that occur at rapid rotation speeds, i.e. taking $\omega \sim V^2/\Omega r^2 \ll V/r \ll \Omega \ll a/r \lesssim g/a$, but *not* making any short-wavelength approximations, (2.13) simplifies to

$$[r\rho V^2k^2\hat{u}_r'/(k^2+n^2)]' + rF_s\hat{u}_r = 0, \tag{4.3}$$

where

$$\rho^{-1}F_s = \frac{4\Omega^2\omega^2n^2}{V^2k^2(k^2+n^2)} - \frac{2\Omega\omega k}{k^2+n^2}\left\{\frac{2k^2}{r(k^2+n^2)} + \left(1+\frac{2n^2}{k^2}\right)\left(\frac{g}{a^2} - \frac{2}{r}\right)\right\}$$
$$- V^2k^2\left[1 - \frac{(3n^2+k^2)}{r^2(k^2+n^2)^2}\right] - V^2\left[\left\{\log\left(\frac{B}{\rho r}\right)\right\}' + \frac{2}{r} - \frac{g}{a^2}\right]\left[\frac{g}{a^2} - \frac{2}{r} + \frac{2k^2}{r(k^2+n^2)}\right]. \tag{4.4}$$

We multiply (4.3) by $\tilde{u}_r$, the complex conjugate of $\hat{u}_r$, and integrate between cylindrical boundaries $r = r_1$ and $r = r_2$ (at which $\tilde{u}_r$ must vanish) to obtain

$$\int_{r_1}^{r_2} \left[\frac{r\rho V^2 k^2 |\hat{u}_r'|^2}{k^2+n^2} - rF_s|\hat{u}_r|^2\right] dr = 0. \tag{4.5}$$

Writing $\omega = \omega_R + i\omega_I$ and equating the imaginary part of (4.5) to zero we find

$$\omega_I \int_{r_1}^{r_2} \rho r |\hat{u}_r|^2 \left[\frac{4\Omega^2 \omega_R n^2}{V^2 k^2} - \Omega k \left\{\frac{2k^2}{r(k^2+n^2)} + \left(1+\frac{2n^2}{k^2}\right)\left(\frac{g}{a^2}-\frac{2}{r}\right)\right\}\right] dr = 0 \tag{4.6}$$

and at once conclude that if $r > 2a^2/g$ throughout the interval $r_1 < r < r_2$, i.e. if the whole fluid lies outside the critical radius, then any amplifying slow mode must have $\Omega k \omega_R > 0$, i.e. must propagate eastwards, for if this were not so the integral could not possibly vanish and we should have a contradiction.

It is interesting (but, when one puts in values for the sun, of lesser importance, since r_c is about one-fifth of the solar radius) that it does not seem possible to make such a strong statement if the whole fluid lies inside the critical radius. It is easy to show from (4.6) that amplifying slow modes with k/n such that

$$r_2 < \frac{(2+3k^2/n^2)}{(1+k^2/n^2)(2+k^2/n^2)}\left(\frac{2a^2}{g}\right) \tag{4.7}$$

is satisfied propagate westwards, but no matter how small r_2 is compared with the critical radius $2a^2/g$ we can always find modes with sufficiently large k^2/n^2 that they will not satisfy (4.7). It is nevertheless noteworthy that k/n does not have to be very small before, for all practical purposes, (4.7) may simply be taken to read $r_2 < 2a^2/g$. This is because on expanding the right-hand side in powers of k^2/n^2 it approximates to $(1-\frac{1}{2}k^4/n^4)\,2a^2/g$ and differs from $2a^2/g$ only as the *fourth* power of k/n.

5. Effects of Ohmic diffusion on the wavy instabilities

A weakness of the preceding theory is that if the magnetic field gradient is gradually changed so as to promote instability the mode which is first to amplify spontaneously has, according to (4.2), an infinite value of n, i.e. zero wavelength in the z direction! As may readily be imagined, no such difficulty arises if the effects of Ohmic dissipation due to a finite electrical conductivity σ are taken into account. Equation (2.2) must then be modified by the inclusion of a term $\eta\nabla^2\mathbf{B}$ on its right-hand side, where the magnetic diffusivity η is defined as $(\sigma\mu)^{-1}$. At rapid rotation speeds satisfying (1.5) the amplifying waves (4.1) have such low frequencies that the acceleration term $\partial\mathbf{u}/\partial t$ in the momentum equation (2.1) is negligible (by a factor of order $V^2/\Omega^2 r^2$) compared with the others. Time dependence thus enters explicitly only in the induction equation (2.2), and this allows us to modify (4.1) by inspection to include the effects of Ohmic diffusion:

$$\omega = \left(\frac{g}{a^2}-\frac{2}{r}\right)\frac{V^2 k}{2\Omega} - i\eta(l^2+n^2) \pm \frac{ikV^2}{2\Omega}\left\{\left(\frac{2}{r}-\frac{g}{a^2}\right)\left[\log\left(\frac{B}{\rho r}\right)\right]' - \left(1+\frac{l^2}{n^2}\right)k^2\right\}^{\frac{1}{2}}, \tag{5.1}$$

bearing in mind that we have taken both l and n to be much larger than r^{-1} but k to be (Or^{-1}), The wave is then marginally stable when

$$r^2\left(\frac{2}{r}-\frac{g}{a^2}\right)\left[\log\left(\frac{B}{\rho r}\right)\right]' = \left(1+\frac{l^2}{n^2}\right)m^2 + \frac{4\Omega^2\eta^2 r^4(l^2+n^2)^2}{m^2 V^4}. \tag{5.2}$$

The right-hand side clearly takes its least value when l is as small as possible, and when the fluid is confined between two narrowly spaced cylinders this will be π/d, where d is the gap width. It is convenient to introduce the parameter

$$\mathscr{C} \equiv \frac{V^2}{2\Omega\eta} \times \frac{d^2}{r^2}, \tag{5.3}$$

which provides an inverse measure of the importance of Ohmic diffusion. While the parameter $V^2/2\Omega\eta$ usually plays this role (see, for example, Acheson & Hide 1973), the short radial and axial wavelengths here increase the ratio of Ohmic decay rate to slow wave frequency [the latter being independent of d; see (5.1)], so that it needs to be weighted by the factor d^2/r^2. We may formally minimize the right-hand side of (5.2) by differentiation to obtain, apparently, the critical mode

$$n^2 = \pi^2/2d^2, \quad m^2 = \pi^2\sqrt{3}/2\mathscr{C} \tag{5.4}$$

and the corresponding critical value of the instability parameter

$$r^2\left(\frac{2}{r} - \frac{g}{a^2}\right)\left[\log\left(\frac{B}{\rho r}\right)\right]' = \frac{3\pi^2\sqrt{3}}{\mathscr{C}}. \tag{5.5}$$

The procedure is, however, a little crude in that the azimuthal wavenumber m ought really to be an integer. When $\mathscr{C} \ll 1$, which corresponds to rather strong Ohmic damping, little error should be expected if we simply take the nearest integer to the value given by (5.4) as the 'true' value of m, and (5.5) indicates that instability will then occur only for magnetic field gradients far steeper than those which would be required according to the diffusionless theory. Even with $\mathscr{C}$ of order unity (5.4) and (5.5) probably remain valid as rough order-of-magnitude guides.

In the weakly diffusive regime $\mathscr{C} \gg 1$, however, the above procedure attributes to m a physically impossible low value (which is not properly interpreted as zero: axisymmetric disturbances are far less readily excited, as § 3 showed), and further investigation reveals that the correct procedure then is to take $m = 1$ and minimize by differentiation with respect to n. The critical mode thus obtained is

$$n^2 = \left(\frac{\mathscr{C}^2}{2\pi^4}\right)^{\frac{1}{3}} \frac{\pi^2}{d^2}, \quad m = 1 \tag{5.6}$$

and the instability criterion essentially becomes

$$r^2\left(\frac{2}{r} - \frac{g}{a^2}\right)\left[\log\left(\frac{B}{\rho r}\right)\right]' > 1. \tag{5.7}$$

The latter is independent of η, so that weak Ohmic diffusion such that $\mathscr{C} \gg 1$ serves only to keep the wavelength in the z direction of the critical mode at a finite, though small, value.

6. Discussion

We shall confine our concluding remarks mainly to the bearing which the above results have on the stability of the local plane-layer model discussed in the introduction and by Gilman (1970). This has the merit of removing curvature effects (it is equivalent to being located well outside the critical radius in the cylindrical model) and highlights magnetic buoyancy as the instability mechanism. Either by modifying (4.1) appropriately or by investigating directly Gilman's equation (21) in the 'slow wave' limit, it is clear that a decrease of B/ρ with height is the essential requirement for instability at rapid rotation speeds in the plane-layer case. Eastward-propagating waves of azimuthal wavelength λ somewhat greater than a scale height then amplify with growth rates ($\sim V^2/\Omega\lambda^2$) which decrease with Ω. The instability can nevertheless be actually cut off at a finite large rotation speed only if Ohmic dissipation is included in the model, as may be seen by letting $r \to \infty$ (with $k \equiv m/r$ finite) in (5.4) and (5.5).

A recent study of the plane-layer problem by Roberts & Stewartson (1977) leads us to expect that diffusive effects may play a more subtle role in our system if in addition to taking $\eta \neq 0$ we take the thermal diffusivity $\kappa \neq \infty$. This has the effect of bringing back into play, to an extent that depends on κ^{-1}, the *conventional* buoyancy forces that arise in a compressible fluid from a vertical entropy gradient, and which in the case of an isothermal atmosphere can be very significant indeed. Roberts & Stewartson show that a layer with $B \propto \rho^{\frac{1}{2}}$ (constant Alfvén speed), though stable in the rapidly rotating regime (1.5) according to the theory with $\eta = 0$ and $\kappa = \infty$, may nevertheless be subject to a 'conductive' instability when the product $\kappa\eta$ is sufficiently small. Low-frequency ($\omega \sim V^2/\Omega\lambda^2$) waves then amplify on the Ohmic time scale.†

We finally note that in the other extreme of *no* conductive or radiative heat transfer, i.e. $\kappa = 0$, the stability equation (2.21) undergoes simple modification. One needs only to replace a by the (local) *adiabatic* sound speed, redefine c accordingly, add to A_2

$$(g+2V^2/r)\,a^2[\log(p\rho^{-\gamma})]'$$

and add to A_4

$$V^2k^2ga^2[\log(p\rho^{-\gamma})]',$$

where γ is the ratio of specific heats. This emphasizes, it seems to us, how *only* by virtue of an entropy gradient in the basic state, which will depend on the radial temperature distribution $T(r)$, may the simplifying assumption $\kappa = \infty$ lead to results atypical of more weakly conductive systems. While we therefore hope that some of our conclusions about magnetic buoyancy apply to the upper layers of the sun, where a more or less adiabatic temperature gradient may reasonably be anticipated, this can of course be decided only after further work.

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† *Note added in proof.* The relationship between the theory in the present paper and that of Roberts & Stewartson (1977) has now been clarified by some more work on magnetic buoyancy instabilities (Acheson 1978).

Appendix A. Magnetic buoyancy in a non-rotating horizontal layer

Consider a horizontal isothermal layer under gravity with $\kappa = \infty$, $\eta = 0$ and a horizontal magnetic field varying in magnitude with height z. Imagine that a flux tube of cross-sectional area Δ at a height z is lifted, without being twisted, to $z+dz$. Let any of its properties ϕ change to $\phi+\delta\phi$, and let the local value of ϕ at the height $z+dz$ be $\phi+d\phi$. Since the mass per unit length $\rho\Delta$ and magnetic flux $B\Delta$ of the tube are conserved, the quantity B/ρ is conserved and

$$\frac{B+\delta B}{\rho+\delta\rho} = \frac{B}{\rho}, \quad \text{i.e.} \quad \frac{\delta B}{B} = \frac{\delta\rho}{\rho}. \tag{A 1}$$

For the tube to be in mechanical equilibrium with its new surroundings its total pressure $p+\frac{1}{2}\mu^{-1}B^2$ must adjust to the local value, whence (to first order)

$$\delta p + B\,\delta B/\mu = dp + B\,dB/\mu. \tag{A 2}$$

Using (A 1) and the fact that $p = a^2\rho$, where $a^2 = RT$ is constant, we may write (A 2) as

$$(a^2+V^2)\,\delta\rho = dp + B\,dB/\mu. \tag{A 3}$$

If $\delta\rho < d\rho$ the tube is lighter than its new surroundings and continues to rise, so the condition for instability is

$$dp + B\,dB/\mu < (a^2+V^2)\,d\rho. \tag{A 4}$$

On using again $p = a^2\rho$ and dividing by dz, we obtain

$$-\frac{d}{dz}\left(\frac{B}{\rho}\right) > 0. \tag{A 5}$$

Appendix B. Energetics of axisymmetric instability

Consider the interchange of two flux rings, denoting the volume and cross-sectional area of either at a radius r by $\xi(r)$ and $\Delta(r)$ respectively. They are initially at radii $r = r_0$ and $r = r_0+dr$, and before the interchange the fluid rotates with angular velocity $\Omega(r)$ (a slight generalization of the case considered in the text) and has constant temperature T. As discussed in the main text, the temperature, angular momentum, magnetic flux and mass of each ring are conserved during the interchange. Further, each ring must be made to occupy the volume vacated by the other (for our interchange argument, which pretends that no other change in the system has taken place, could surely have no relevance otherwise), and this can be arranged subject to the above constraints only if, having picked arbitrarily the initial volume of the first ring, the initial volume of the second ring is chosen in a specific way, which we determine in the following subsection.

Consider any quantity ϕ (e.g. density) pertaining to a fluid ring. After the interchange denote the new value of ϕ for the ring that was at r_0 by ϕ_A and denote the new value of ϕ for the ring that was at r_0+dr by ϕ_B. Also denote $\phi(r_0)$ by ϕ_0 and denote $\phi(r_0+dr)$ by ϕ_0^+.

Relationship between the initial volumes of the rings

Each occupies the volume vacated by the other, so

$$\xi_0 = 2\pi r_0 \Delta_B, \quad \xi_0^+ = 2\pi(r_0+dr)\Delta_A. \tag{B 1}$$

By conservation of mass

$$\rho_0\xi_0 = \rho_A\xi_0^+, \quad \rho_0^+\xi_0^+ = \rho_B\xi_0, \tag{B 2}$$

while by conservation of magnetic flux

$$B_0\Delta_0 = B_A\Delta_A, \quad B_0^+\Delta_0^+ = B_B\Delta_B. \tag{B 3}$$

By suitably combining (B 1)–(B 3) we obtain

$$\frac{B_0^+}{B_B} = \frac{B_A}{B_0} = \frac{\xi_0}{\xi_0^+}\left(1+\frac{dr}{r_0}\right). \tag{B 4}$$

Since the ring originally at $r = r_0$ must come into mechanical equilibrium with its new surroundings and the fluid is isothermal with sound speed a,

$$\begin{aligned} a^2\rho_0^+ + \frac{B_0^{+2}}{2\mu} &= a^2\rho_A + \frac{B_A^2}{2\mu} \\ &= a^2\rho_0\frac{\xi_0}{\xi_0^+} + \frac{B_0^2}{2\mu}\left(\frac{\xi_0}{\xi_0^+}\right)^2\left(1+\frac{2dr}{r_0}\right), \end{aligned} \tag{B 5}$$

where we are working correct to first order in dr/r_0. Expanding ρ_0^+, B_0^+ and ξ_0^+ in Taylor series correct to that order (i.e. $\rho_0^+ = \rho_0 + \rho'\,dr$, etc., where the prime denotes differentiation with respect to r) and equating first-order terms in (B 5) we obtain, on finally using the magnetohydrostatic constraint on the basic state (2.7),

$$\frac{\xi'}{\xi_0} = \frac{2B_0^2/\mu r_0 + \rho_0 g}{B_0^2/\mu + a^2\rho_0}. \tag{B 6}$$

This tells us how the initial volume $\xi_0 + \xi'\,dr$ of the ring initially at $r = r_0 + dr$ must be chosen in order that each ring occupies the volume vacated by the other.

Change in rotational kinetic energy

Since angular momentum and mass are conserved by an individual ring, so is the quantity Ωr^2. In an analogous way to the derivation of (B 4) from (B 2) and (B 3) we thus obtain

$$\frac{\Omega_B}{\Omega_0^+} = \frac{\Omega_0}{\Omega_A} = \left(1+\frac{dr}{r_0}\right)^2. \tag{B 7}$$

The net change in rotational kinetic energy is given by

$$\delta_R = \tfrac{1}{2}(\rho_B\Omega_B^2 - \rho_0\Omega_0^2)\,\xi_0 r_0^2 + \tfrac{1}{2}(\rho_A\Omega_A^2 - \rho_0^+\Omega_0^{+2})\,\xi_0^+(r_0+dr)^2, \tag{B 8}$$

and after using (B 2) and (B 7) to eliminate quantities with suffixes A or B, expanding quantities such as ξ_0^+ in a Taylor series correct to order $(dr/r_0)^2$, and finally using (B 6), we obtain

$$\frac{\delta_R}{\rho_0\xi_0(dr)^2} = \frac{2\Omega}{r}\frac{d}{dr}(\Omega r^2) - \frac{\Omega^2 r V^2}{a^2+V^2}\frac{d}{dr}\left[\log\left(\frac{B}{\rho r}\right)\right]. \tag{B 9}$$

Potential energy change

The net change in gravitational potential energy is given by

$$\delta_G = g^*(dr)\,(\rho_0\xi_0 - \rho_0^+\xi_0^+), \tag{B 10}$$

and expanding this and applying (B 6) to eliminate ξ' then gives

$$\frac{\delta_G}{\rho_0\xi_0(dr)^2} = \frac{g^* V^2}{a^2+V^2}\frac{d}{dr}\left[\log\left(\frac{B}{\rho r}\right)\right]. \tag{B 11}$$

Note that we have presented the results for kinetic and potential energy a little differently in the main text, where it is convenient to think in terms of the 'effective' gravity $g \equiv g^* - \Omega^2 r$. Essentially, from this viewpoint, the second term of (B 9) is reckoned as potential rather than kinetic energy and put into (B 11) instead.

Magnetic energy change

The magnetic energy per unit volume is $B^2/2\mu$, so the net change in magnetic energy is

$$\delta_M = \frac{\xi_0^+}{2\mu}(B_A^2 - B_0^{+2}) + \frac{\xi_0}{2\mu}(B_B^2 - B_0^2). \tag{B 12}$$

Following the procedures of the previous two subsections, first using (B 4), we obtain

$$\frac{\delta_M}{\rho_0\xi_0(dr)^2} = \frac{a^2V^2}{(a^2+V^2)^2}\left(g - \frac{2a^2}{r}\right)\frac{d}{dr}\left[\log\left(\frac{B}{\rho r}\right)\right]. \tag{B 13}$$

Internal energy change

The elastic energy e per unit mass stored in a gas owing to compression is, to within an unimportant additive constant, $-\int p\,d(1/\rho)$. For our isothermal case this is therefore $a^2\log\rho$. Using (B 2), the change in internal energy

$$\delta_I = \xi_0^+(\rho_A e_A - \rho_0^+ e_0^+) + \xi_0(\rho_B e_B - \rho_0 e_0) \tag{B 14}$$

can thus be expressed succinctly as

$$\delta_I = a^2(\rho_0^+\xi_0^+ - \rho_0\xi_0)\log(\xi_0^+/\xi_0), \tag{B 15}$$

and by proceeding further as before we find

$$\frac{\delta_I}{\rho_0\xi_0(dr)^2} = \frac{-a^2V^2}{(a^2+V^2)^2}\left(g + \frac{2V^2}{r}\right)\frac{d}{dr}\left[\log\left(\frac{B}{\rho r}\right)\right]. \tag{B 16}$$

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